

DIRECTIONAL SPECTRUM OF OCEAN WAVES

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ABSTRACT

This paper describes a methodology for obtaining the directional spectrum of ocean waves from time series measurement of wave elevation at several gauges arranged in linear or polygonal arrays. Results of simulated studies using sinusoidal wave fields are described. For these sinusoidal wave fields, analysis of both linear and polygonal arrays consistently yielded accurate directions, in contrast with the limited success of Pawka (1974) in case of linear arrays and Esteva (1976, 1977) in case of polygonal arrays. An attempt is made to determine array location and configuration for the coast off Calangute, Goa.

INTRODUCTION

Knowledge of the direction of wave approach is very crucial in the design of harbours and coastal structures (Pawka, 1974). One of the best methods in the study of littoral transport is through directional spectrum (Komar and Inmam, 1970).

In conventional spectral analysis, to every discrete frequency band one assigns a power spectral density. In directional spectrum, to every frequency band one assigns not only a power spectral density but also a direction wherefrom waves in this band come.

The direction can be determined from either of two types of measurements. In the first type, time series measurement of several parameters is made simultaneously at a single location, e.g. the pitch-roll buoy of Hasselmann et al (1973) which measures vertical acceleration, geographic (compass) orientation of the buoy, and two components of wave slope.

In the second type of measurement - described in this paper - time series measurement of a single parameter, viz. wave elevation is made simultaneously at several gauges which may be arranged in linear or polygonal arrays.

In this paper, using cross-spectrum analysis, we first show how to determine phase difference ϕ_k between the arrival of a wave crest at two gauges, for certain discrete frequencies f_k from the time series

of wave elevation simultaneously obtained at the two gauges. Next, at these frequencies, wave directions α_k are deduced from the phase differences ϕ_k .

Results of simulated studies using sinusoidal wave fields are described. An attempt is made to determine the array location and configuration for the coast off Calangute, Goa.

LINEAR ARRAY (2-gauge)

Phase

Let $z_k = x_k + i y_k$, $k=0,1,\dots,N-1$, be a complex time series, where $i = \sqrt{-1}$, and x_k , y_k are time series elevations obtained simultaneously at two gauges. Let the sampling interval be h . The Fourier transform Z_k of the series z_k is defined by

$$Z_k = \sum_{n=0}^{N-1} z_n \exp(-2\pi i k n / N), k=0,1,\dots,N-1$$

at frequencies $f_k = k/Nh$. Literature suggests that the phase differences ϕ_k at frequencies f_k may be determined from Z_k by the relation

$$\phi_k = \arctan(-\text{Im} Z_k / \text{Real } Z_k)$$

Not being successful in determining the phase difference by the above formula we had perforce to resort to a bit round about method involving intermediate computation of Fourier transforms X_k , Y_k of the series x_k and y_k respectively.

The Fourier transforms are given by (see Bendat and Piersol, 1971):

$$X_k = (Z_k + Z_{N-k}^*)/2$$

$$Y_k = (Z_k - Z_{N-k}^*)/(2i)$$

where Z^* is the complex conjugate of Z . The phases, at frequencies f_k , for the time series x , y are given by

$$PX_k = \arctan(-\text{Im} X_k / \text{Real } X_k)$$

$$PY_k = \arctan(-\text{Im} Y_k / \text{Real } Y_k)$$

By taking into consideration the signs of the numerator and the denominator, PX_k and PY_k are uniquely determined in the interval $(-\pi, \pi)$.

Finally the phase difference ϕ_k between the arrival of a wave crest at the two gauges is uniquely determined from PX_k and PY_k in the range $(-\pi, \pi)$

$$\phi_k = PX_k - PY_k$$

It may be noted that phase difference ϕ_k has physical significance only for frequencies $f_k = k/(Nh)$, in the range defined by, $0 \leq k \leq N/2$, the high frequency limit being the Nyquist frequency.

We also note that the cross-spectrum analysis does not determine the exact phase difference. What it does determine is a value in the interval $(-\pi, \pi)$ reduced from the true phase difference by application of the definition of angle in co-ordinate geometry which states that θ and $+2n\pi + \theta$ are equivalent, where n is an integer and θ is any arbitrary angle. In reality these phase differences may refer to quite different situations. For example consider a sinusoidal long crested wave of period T sec approaching two points A and B. A phase difference of 30° may be defined to mean that a crest will arrive at B, $T/12$ sec after it crosses A. By this definition a phase difference -330° signifies that the wave crest arrives at B, $(11T)/12$ sec before it crosses A. Surely these are two distinct situations physically, unfortunately rendered indistinguishable by cross-spectrum analysis. It therefore stands to reason that the phases determined by the

cross-spectrum analysis will faithfully represent the physical situation only if we are able to restrict the phase difference in the physical situation also to the interval $(-\pi, \pi)$. This is obviously ensured by restricting the distance between the points A and B to less than half the wavelength at this period.

Direction

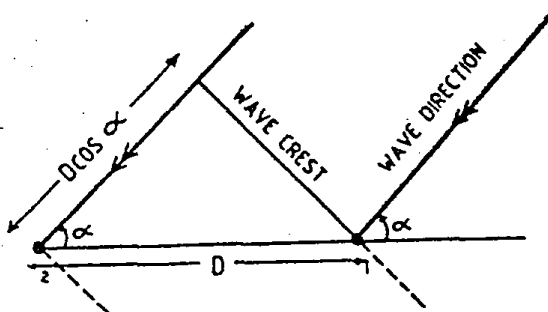
We shall now establish the relationship between wave direction α_k and phase lag ϕ_k .

Let $f = f_k$, for some k , $0 \leq k \leq N/2$. Dropping subscripts in the following consider a sinusoidal long crested wave of frequency f (i.e. period T , $T = 1/f$) moving across two gauges separated by distance D , $D < \lambda/2$, where λ is the wavelength. Let τ be the time difference between the arrival of a wave crest at the two gauges. Then the wave direction α , the angle made by the wave with the line joining the gauges (see Fig. 1), can be determined directly from the equation

$$D \cos \alpha = c \tau \quad \dots\dots\dots 1$$

as the wave celerity c is a function of f alone.

FIGURE-1



Thus for specified D and f , Equation (1) establishes a relationship for determining α from time difference τ .

We shall now show how to evaluate c as a function of f using linear wave theory, and then show how τ is evaluated from phase difference ϕ .

The wave number k , corresponding to frequency f is first determined from the following well known formula of linear wave theory, by the method of iteration.

$$(2\pi f)^2 = kg \tanh kd$$

where d , the mean water depth may be assumed constant at the gauge sites.

The wave length λ , $\lambda = 2\pi/k$, can then be determined and the celerity, $c = \lambda/T = \lambda f$ evaluated.

For time difference T , the phase difference is 2π , so the phase difference ϕ corresponding to time difference τ must satisfy

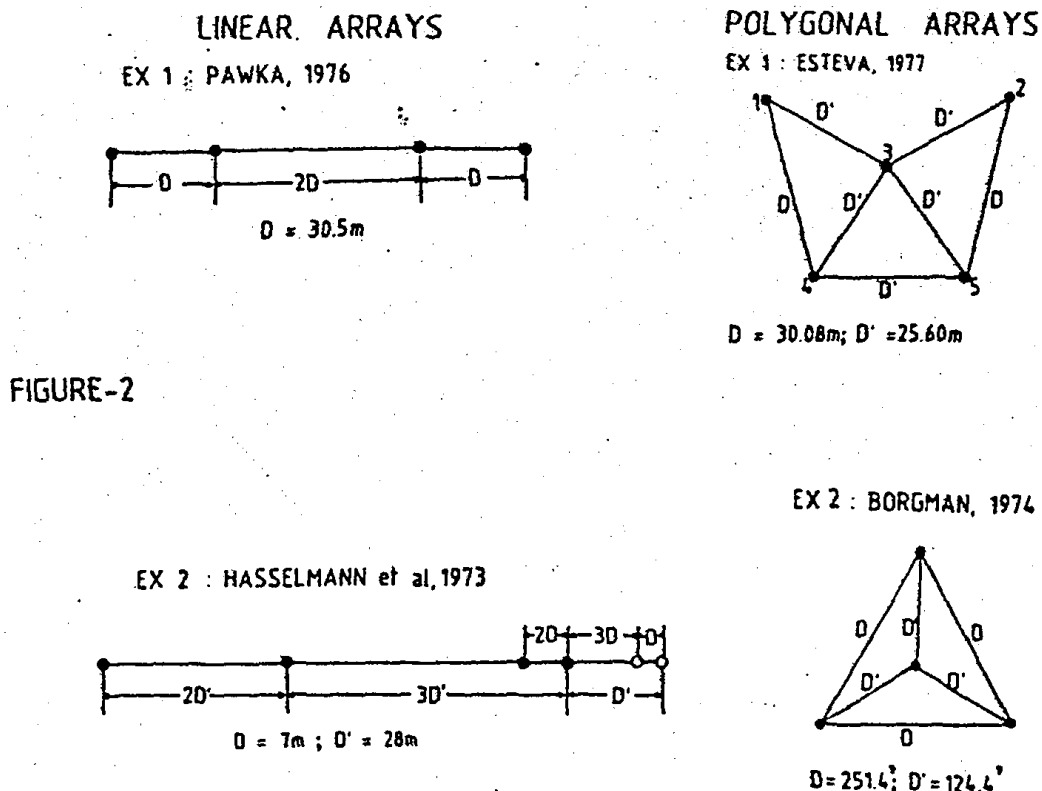
$$\tau = \phi \cdot T / (2\pi)$$

Having evaluated c and τ , α can be determined from Equation (1). We note that if ϕ is positive α is acute and if ϕ is negative, α is obtuse. But we are unable to distinguish waves arriving at the gauges at the same angle but from opposite sides, i.e. at angles α and $2\pi - \alpha$ (see Fig 1).

However, oftentimes it may be possible to assume that waves come from just one side, the seaward side, e.g. swell at a long coastline. If such be the case then Equation (1) unambiguously determines

the direction α provided that our assumption that $D < \lambda/2$ holds.

At this stage one may conclude that should our interest lie in determining directions for waves of period larger than, say, 5 sec, then a single pair of gauges installed at 10 m depth at a distance D apart, $D < 18.295 \text{ m} = \lambda/2$, should suffice. But because of another important consideration, namely, accuracy and resolution, in practice, several gauges are installed in a linear array with gauge separations generally $D, 2D, 3D, \dots$, etc, the smaller separations determining directions of waves of higher period and vice versa (see Fig 2).



An inherent assumption is that the mean water depth d is nearly constant over an area encompassing the gauges. This is easily satisfied by proper choice of site for the gauges.

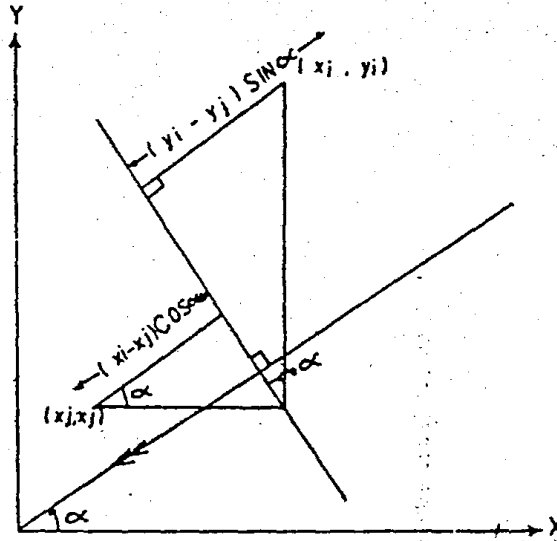
POLYGONAL ARRAY (3-gauge)

More often than not it is likely that waves come from all sides, for example in cases of sea and/or swell at locations far from the shore or near promontories. But as noted earlier, linear arrays are not able to distinguish between waves arriving at the same angle but from opposite sides (this defect is sometimes referred to as an ambiguity within a mirror-symmetry). For such cases therefore linear arrays fail in correctly determining directions. We shall show how polygonal arrays may be used for precisely such cases.

Let (x_i, y_i) and (x_j, y_j) be the positions of two gauges, referred to convenient axes. Let α_j be the angle made by the wave with

the positive direction of the x-axis, the angle being measured anticlockwise (see Fig.3).

FIGURE-3



For length λ , the phase difference is 2π . Therefore for length $(x_i - x_j) \cos \alpha + (y_i - y_j) \sin \alpha$, the corresponding phase difference is,

$$\phi_{ij} = (2\pi/\lambda) [(x_i - x_j) \cos \alpha + (y_i - y_j) \sin \alpha]$$

$$= k [(x_i - x_j) \cos \alpha + (y_i - y_j) \sin \alpha]$$

In particular,

$$\phi_{12} = k [(x_1 - x_2) \cos \alpha + (y_1 - y_2) \sin \alpha] \quad \dots\dots\dots 2$$

$$\phi_{13} = k [(x_1 - x_3) \cos \alpha + (y_1 - y_3) \sin \alpha] \quad \dots\dots\dots 3$$

From Equations (2), (3) it is clear that

$$\sin \alpha = [(x_1 - x_2) \phi_{13} - (x_1 - x_3) \phi_{12}] / P$$

$$\cos \alpha = [(y_1 - y_3) \phi_{12} - (y_1 - y_2) \phi_{13}] / P$$

where, $P = k [(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)]$, does not vanish for nonlinear arrays.

Then α is uniquely determined from the following equation by considering the signs of the numerator and denominator,

$$\alpha = \arctan \left\{ \frac{[(x_1 - x_2) \phi_{13} - (x_1 - x_3) \phi_{12}] / \text{sgn } P}{[(y_1 - y_3) \phi_{12} - (y_1 - y_2) \phi_{13}] / \text{sgn } P} \right\}$$

where $\text{sgn } P = 1$, for $P > 0$ and $\text{sgn } P = -1$, for $P < 0$

Clearly the direction α determined above will be correct only if D_{12} and D_{13} are less than $\lambda/2$, where λ , is the wavelength at the spectral frequency under consideration and D_{12} , D_{13} are the distances between gauges 1-2 and 1-3 respectively.

Design and interpretation

Figure 4 is a plot between the wave period T and the maximum gauge separation permissible, i.e. half a wavelength which as we have already seen ensures that phases determined by the cross-spectrum

analysis faithfully represent the physical situation. The plot is valid for a depth $d = 10$ m. This plot may be made to serve two purposes. One, as a tool in designing the array to suit local conditions of bathymetry and wave climate, meaning thereby, the determination of optimum (i.e. least) number of gauges and their separations. And the other purpose, equally important, being for the interpretation of directional information obtained as a result of the analysis, i.e. identifying meaningful directional information.

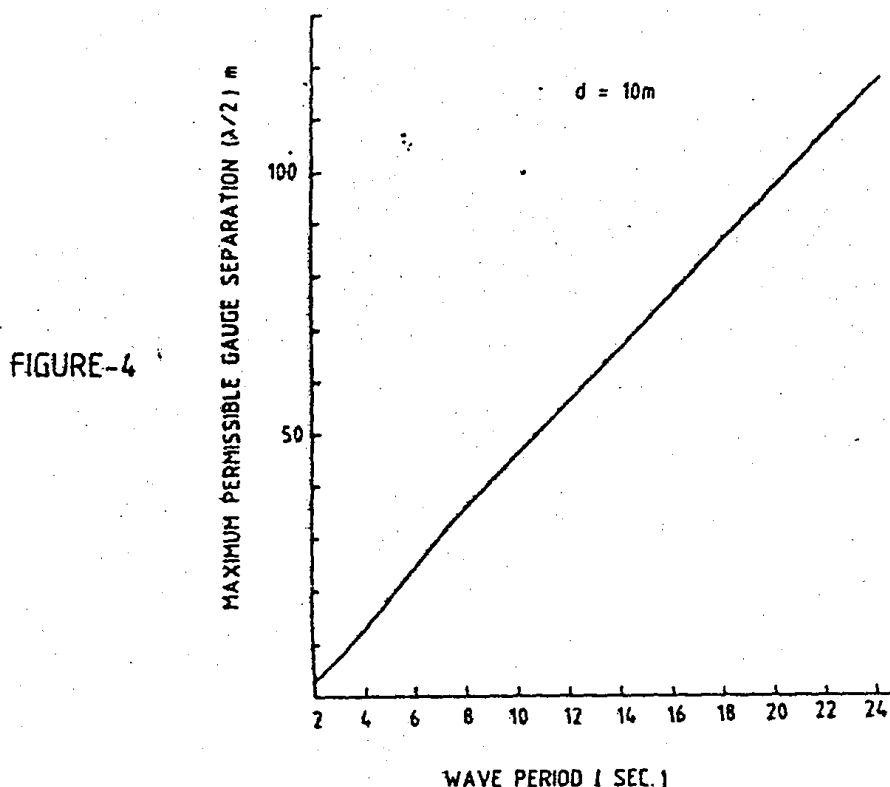
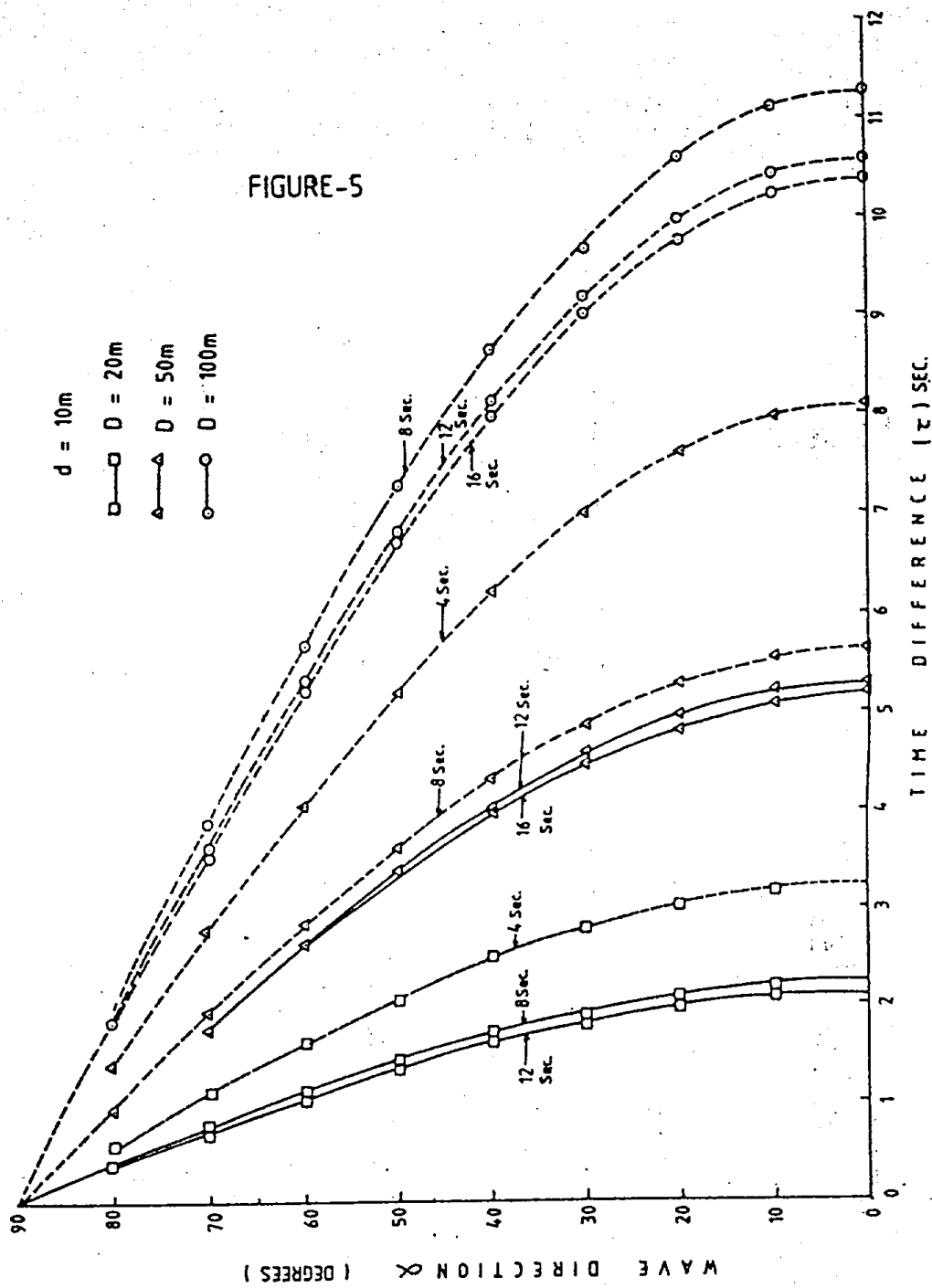


Fig. 5 shows time difference plotted against wave direction for wave periods 4 sec, 8 sec, 12 sec and 16 sec at three gauge separations $D = 20$ m, 50 m and 100 m. The dashed curves signify that the criterion $D < \lambda/2$ is not valid as the time difference τ is not less than $T/2$. We note that we cannot use gauge separation $D = 100$ for determining directions in the 4 cases of wave periods considered. In case of gauge separation $D = 50$, directional information is valid for $T = 12$ and 16, but invalid for $T = 4$ and 8. And in case of $D = 20$, the directional information is invalid only for $T = 4$. We thus see that in case of period $T = 12$, directional information is correct for gauge separations 20 and 50, and a further perusal of the same figure tells us that for gauge separation $D = 50$, angles are better resolved, specially for $\alpha < 40^\circ$. This is clearly brought home from the fact that time difference obtained from directional analysis will be accurate at best to within $\pm h$, the sampling interval of the time series elevation, the least and most ambitious value of h being 0.25 sec.



SIMULATION STUDIES

Wave Simulation

Let (x, y) be the position of a gauge referred to convenient axes, and α , the wave direction reckoned positive anticlockwise (See Figure 6). Then it may be seen that the wave elevation, ξ as a function of time, t at the gauge is given by

$$\xi = A \cos [2\pi f t + \theta + \kappa (x \cos \alpha + y \sin \alpha)]$$

where f is the wave frequency, κ is its corresponding wave number and θ is the initial phase at the origin and A is the amplitude.

FIGURE-6

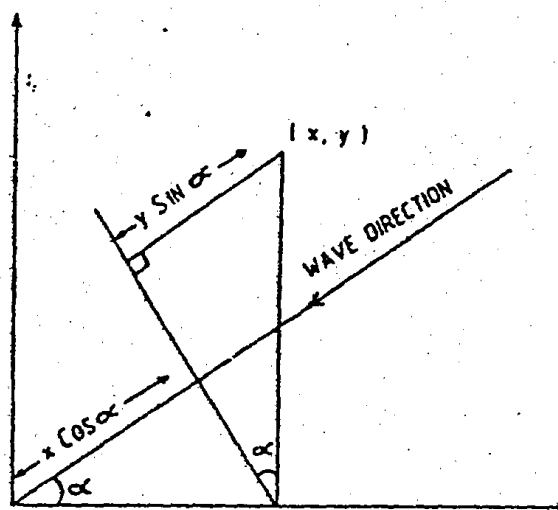


TABLE 1

	Closest spectral		Simulated wave train			
	Line (k)	Period (sec)	Amplitude	Period* (sec)	Phase+ (deg)	Direction (deg)
Case (1)	61	16.79	11.43	16.75	15	120
	64	16.00	15.24	15.97	25	120
	67	15.28	11.43	15.28	20	120
Case (2)	125	8.19	11.43	8.18	15	340
	128	8.00	15.24	7.99	25	340
	131	7.82	11.43	7.81	20	340

* Indicates approx. period

+ Indicates initial phase at the origin

The above equation was used to simulate two cases of sinusoidal wave fields at five gauges located at $A=(-24.385, 7.92)$, $B=(-15.04, -20.73)$, $C=(0., 0.)$, $D=(-15.04, -20.73)$, and $E=(15.04, -20.73)$, which are in fact gauge numbers 1 through 5 of Esteva (1977) (see Figs 2, 7). Each simulated wave field was constructed from three sinusoidal

frequencies $3/1024$ Hz apart from each other, each frequency 0.134 Hz removed from the closest spectral frequency in lines with idem. Sampling interval h and number of points N in the time series were taken as 1 sec and 1024 respectively. The characteristics of the two cases of wave fields generated are given in Table 1. They may be briefly described as 16 sec swell coming from the second quadrant and 8 sec swell coming from the fourth quadrant respectively.

From the time series thus generated we recover directional information by use of both types of analysis — 2-gauge as well as 3-gauge.

Analysis of simulated wave fields

The first step was to determine phase information PX_k and PY_k at the five gauges from the cross-spectrum analysis of gauge pairs 1-2, 1-3, and 4-5. This was done using an off-the-shelf "user-friendly" computer program CROSS-SPEC1 developed by Fernandes and Gouveia (1986). We chose the Hanning window in time domain for reducing spillover of energy so as to mimic Esteva (1977). However spectral densities (amplitude squared) reported in this paper have been scaled up by a factor of $1/0.375$ to compensate for loss of energy due to the window used. From the phase information thus obtained we computed directions for linear and polygonal arrays. We were able to directly handle both 2-gauge and 3-gauge cases simultaneously because of the peculiar configuration of the array, the orientation of the axes and an uniformity in mathematical treatment of the two cases. We can look upon gauge pairs 2-1 and 5-4 as two separate 2-gauge arrays in addition to $C_3 = 10$ possible combinations of 3-gauge (i.e. polygonal) arrays, viz. 1-2-3, 1-2-4, 1-2-5, 1-3-4, 1-3-5, 3-4-5, 3-2-5, 1-4-5, 2-4-5, and 3-2-4.

These ten 3-gauge combinations may be grouped into three distinct classes for the purpose of determination of directions at a specified spectral frequency.

1. Class "YES", composed of 3-gauge arrays all of whose gauge separations are within half a wavelength at the specified frequency, thereby always giving correct directions.

2. Class "IF", composed of those 3-gauge arrays that contain a single gauge separation (i.e. one arm of the triangle) which exceeds half the wavelength at the specified frequency, thereby giving correct directions only if we do not use that particular arm in the computations of phase differences. This can easily be done.

3. Class "NEVER", composed of those 3-gauge arrays that contain at least two arms of the triangle that exceed half the wavelength. In this case directions determined will always be incorrect.

Analysis of 16 sec swell

For case (1), i.e. 16 sec swell we note from Fig 7 that the highest gauge separation in the array of gauges is 48.77 m which is well within half a wavelength ($\lambda/2 = 77.245$ m) at this wave period (see Fig 4). We therefore note that the two 2-gauge arrays and all ten 3-gauge arrays will yield correct directions. Table 2 presents the results of the two 2-gauge and two 3-gauge arrays. We see that the directions obtained are close to the assigned direction. These results are in agreement with Esteva (1976, 1977)

TABLE 2
(16 sec swell from 120°)

Sr No	Spectral Line Period		Computed directions				Spectral Density
			3-1-2	1-4-5	2-1	5-4	
1	58	17.66	119.79	119.98	122.20	122.21	.00
2	59	17.36	120.09	120.11	120.69	121.44	.03
3	60	17.07	119.99	120.00	120.68	120.68	14.23
4	61	16.79	120.00	119.99	120.07	120.07	83.94
5	62	16.52	120.00	120.01	119.59	119.59	33.95
6	63	16.25	120.00	120.01	120.72	120.72	22.57
7	64	16.00	120.01	120.00	120.06	120.06	151.65
8	65	15.75	120.00	119.99	119.50	119.49	57.94
9	66	15.52	120.01	120.01	120.80	120.80	11.46
10	67	15.28	119.99	120.00	120.05	120.05	86.66
11	68	15.06	120.01	120.17	119.09	119.64	31.05
12	69	14.84	119.98	119.92	118.82	118.97	.08
13	70	14.63	119.96	120.23	118.27	118.39	.00

Analysis of 8 sec swell

For case (2), i.e. 8 sec swell coming from 340° the position is a bit complex. For this period the gauge separations for three pairs, viz. 1-2, 1-5 and 2-4 each of which is 48.77 m exceeds half the wavelength ($\lambda/2 = 35.44$ m) at this period (see Figs 7, 4). Therefore in case of 2-gauge arrays, the pair 2-1 will not yield correct directions whereas the pair 5-4 will always give correct directions. In case of 3-gauge arrays, we note that three triads, viz. 3-1-4, 3-4-5 and 3-2-5 belong to the class "YES", four triads, viz. 3-1-2, 4-1-5, 5-2-4 and 3-2-4 belong to the class "IF" and two triads, viz. 1-2-5 and 1-2-4 belong to the class "NEVER". Table 3 gives the results for the 2-gauge pair 5-4 and for some 3-gauge triads belonging to class "YES" or "IF". We note that the 3-gauge arrays faithfully reproduce the assigned direction. We also note that while the specified direction was 340° = -20°, the 2-gauge analysis yields a wave direction of 20°, which clearly demonstrates the stated ambiguity of direction within a mirror-symmetry for linear arrays.

TABLE 3
(8 sec swell from 340°)

Sr No	Spectral Line Period		Computed directions				Spectral Density
			3-1-2	4-1-5	3-4-5	5-4	
1	122	8.39	-20.01	-20.21	-20.08	13.39	.00
2	123	8.33	-19.98	-19.96	-19.99	15.18	.03
3	124	8.26	-20.00	-20.00	-20.00	17.94	14.22
4	125	8.19	-19.97	-20.00	-20.00	19.89	83.93
5	126	8.13	-20.00	-20.00	-20.00	21.04	33.96
6	127	8.06	-19.98	-20.00	-20.00	17.99	22.55
7	128	8.00	-20.00	-20.00	-20.00	20.05	151.66
8	129	7.94	-19.99	-20.00	-20.00	21.02	57.91
9	130	7.88	-20.00	-19.89	-20.00	17.60	11.48
10	131	7.82	-20.06	-20.00	-20.00	19.76	86.65
11	132	7.76	-19.99	-20.00	-20.00	21.17	31.06
12	133	7.70	-20.07	-19.99	-19.96	22.93	.08
13	134	7.64	-19.79	-20.05	-20.09	24.69	.00

The results of 8 sec swell for 3-gauge arrays are at variance with those of Esteva (1977), who surmised, that (1) 3-gauge arrays which are nearly equilateral have greater direction discernability, and (2) that the analysis gives correct directions only for waves coming from certain directions.

We find that his second conclusion is totally erroneous. Specifically we note that all directions α within $(0, 2\pi)$ are faithfully reproduced, restricting the analysis, of course to 3-gauge combinations in the first two classes, viz "YES" and "IF" discussed earlier and being careful about the gauge sequence for the latter class as already indicated.

His first conclusion, i.e. that nearly equilateral arrays have greater direction discernability is close to the truth, but only incidentally so. We note that the nearly equilateral combinations, viz. 1-3-4, 3-4-5 and 3-2-5 all belong to the "YES" class, while the remaining seven 3-gauge arrays belong to the "IF" and "NEVER" classes, which will definitely give erroneous directions for class "NEVER" triads and are liable to error for class "IF" triads if precautions are not taken.

For completeness, we may admit that while we have used a mean water depth $d = 10$ m, Esteva (1977) uses $d = 9.14$ m. However, in spite of this difference, the above 3-gauge classification stands - for the wave periods under consideration.

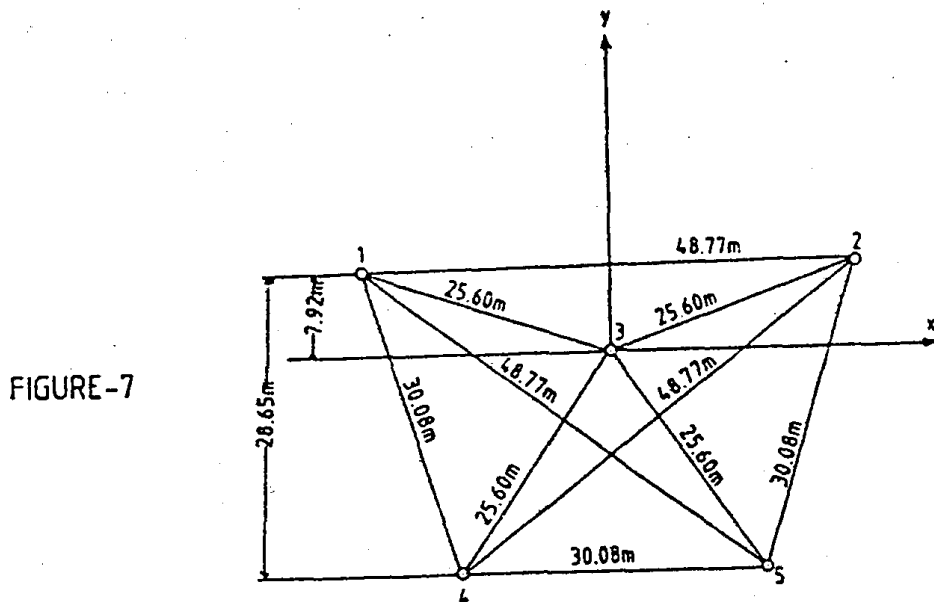


FIGURE-7

DESIGN OF ARRAY FOR CALANGUTE COAST

The coastline at Calangute, Goa is straight and the depth contours nearly parallel to the coast. Moreover around the 10 m contour which lies around 4 km from the shore, the topography is quite smooth. As such a linear array should be adequate for determination of directions. Depending on the requirement, i.e. the range of periods for which directional information is desired we can easily design a linear array.

CONCLUSIONS

An unified treatment of linear and polygonal arrays has been given. For simulated sinusoidal wave fields, analysis of both linear and polygonal arrays consistently yielded directions accurately and unambiguously, in contrast with limited success reported by Pawka (1974) in case of linear arrays and Esteva (1976, 1977) in case of polygonal arrays. The accuracy was achieved by using only gauge pairs with separations less than half a wavelength for determining phase difference(s) required for computation of direction at each spectral frequency. Emphasis is laid on two generally ignored aspects, one, the concept of phase difference and, two, the proper interpretation of directional information derived from gauge arrays.

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